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Finite-Amplitude Deep-Water Waves on Currents

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FINITE-AMPLITUDE DEEP-WATER WAVES ON CURRENTS

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Accurate integral properties of plane periodic deep-water waves of amplitudes up to the steepest are tabulated by Longuet-Higgins (1975). These are used to define an averaged Lagrangian which, following Whitham, is used to describe the properties of slowly varying wave trains. Two examples of waves on large-scale currents are examined in detail. One flow is that of a shearing current, $V(x)\,\boldsymbol{j}$, which causes waves to be refracted. The other flow, $U(x)\,\boldsymbol{i}$, varies in the direction of wave propagation and causes waves to either steepen or become more gentle. Some surprising features are found.

1. Introduction

The interaction between water-waves and currents was first fully analysed by Longuet-Higgins & Stewart in a number of papers culminating in a summary paper, Longuet-Higgins & Stewart (1964). That work is based on the linear solution for water waves. Finite-amplitude water-wave interactions with currents are studied by Crapper (1972). He uses an approximate finite-amplitude water-wave Lagrangian proposed by Lighthill (1967) in Whitham's method of averaging non-linear waves (see Whitham 1974, chs 14 and 15). A recent survey by Peregrine (1976) gives more details of previous work.

Accurate solutions for properties of plane periodic deep-water waves are given by Schwartz (1974) and Longuet-Higgins (1975). (The latter paper is referred to as M.S.L.-H. throughout this paper.) They both use computer assisted evaluation of high-order perturbation series. M.S.L.-H. shows that integral properties of the waves, such as mean energy density, have a maximum for waves of slightly less height than the steepest. The solutions are least accurate for the steepest waves but recent work by Longuet-Higgins & Fox (1978) confirms the accuracy of the results for waves up to the maximum energy. Note each property, such as energy and phase velocity, reaches its maximum at slightly different wave steepnesses.

In the next section we use results of M.S.L.-H. to define an averaged Lagrangian. Following Lighthill (1967) it is taken in the form

$$\mathscr{L} = \rho g f(a^2k^2)/k^2 \tag{1.1}$$

for waves on still water. The wave action and wave-action flux are also determined. This means that when a steady wave train on a current is considered, appropriate equations can be written down (§ 3). It is then only a matter of algebra to find solutions for particular current distributions when the current varies in a single coordinate direction.

Two examples are considered in detail in §§ 4 and 5. The first is a current distribution, $V(x)\,\boldsymbol{j}$, which refracts the propagation direction of the waves. For linear waves a caustic arises if they are refracted to propagate parallel to the current. A related singularity occurs in the finite-amplitude solutions presented here. The other current distribution, $U(x)\,\boldsymbol{i}$, is only studied for waves propagating in the $\boldsymbol{i}$ direction. Thus, the current is either in the same direction as, or opposite to, the wave number vector. A sufficiently 'adverse' current can 'stop' wave propagation. This also corresponds to a caustic for linear waves.

It is shown that these two caustics are examples of fundamentally different types. For the first, refractive, caustic, the waves do not approach maximum steepness and we consider that the actual wave behaviour is quite regular reflexion at the caustic. The second, stopping, caustic, yields solutions that smoothly reach the waves of maximum energy. We consider that this should be taken to represent breaking.

The slowly varying wave theory used does not include any wave reflexion. However, waves are reflected at caustics and properties of reflected waves are discussed. The discussion is based on the conservation of wave-action, which has been demonstrated for a relatively large class of conservative wave systems by Hayes (1970). Thus the reflected wave train is assumed to have the same wave-action flux as the incident wave train, except for a change in sign of the component directed toward a caustic. Solutions, such as those presented here, based on a single wave train cannot be appropriate close to a caustic because of nonlinear interactions between incident and reflected waves, but it is plausible to think in terms of long finite wave trains where such interactions only occur close to a caustic. Unlike linear waves, solutions for reflected nonlinear waves do not necessarily exist.

In a companion paper, Peregrine & Smith (1979), henceforth referred to as P. & S., various approximations to wave propagation near a caustic are discussed. In particular, the above two types of caustic are examined with a near-linear theory. Although much can be learned from near-linear theory there are several ways in which the fully nonlinear solutions of this paper usefully complement and extend the near-linear analysis.

The solutions with wave-action flux having zero component in the x direction are of interest. In particular the solution corresponding to waves 'stopped' on an adverse stream raises the question of how the concept of group velocity may be extended to nonlinear waves. This subject is discussed in § 6 with the conclusion that there are several different velocities which might correspond to group velocity and all have their defects.

2. An averaged Lagrangian for finite-amplitude waves

For plane waves of amplitude a, frequency σ and wave number $\boldsymbol{k}$ on otherwise still deep water an averaged Lagrangian has the form

$$\mathscr{L}(\sigma, \boldsymbol{k}, a).$$

The dispersion relation, $\mathscr{L}_a = 0$, or $G(\sigma, \boldsymbol{k}, a) = 0$,

and the Lagrangian differ from those for waves in finite depth of water in that the influence of the waves on mean currents and the mean water level is not significant.

Lighthill (1965*b*) pointed out that the dispersion relation can be used to eliminate one of the variables from $\mathscr{L}$. Further simplification is possible by considering the physical dimensions of the variables. Lighthill (1967) proposed the Lagrangian

$$\mathscr{L}_1(\sigma, \boldsymbol{k}) = (\rho g/k^2) f(z), \tag{2.1}$$

where $k = |\boldsymbol{k}|$ and $z = (\sigma^2/gk) - 1$. The function $f(z)$ was chosen to be a simple polynomial approximation based on near-linear theory and the wave of greatest height. Crapper (1972) extended the use of this Lagrangian to calculate the behaviour of deep-water waves on a current.

M.S.L.-H. shows that waves just short of the steepest have surprising properties. The steepest waves are neither the most energetic nor the fastest. Most integral properties of periodic water waves have maxima, at differing steepnesses, before the steepest waves are reached. These surprising properties and the availability of accurate properties for steep waves prompted the present work.

For plane periodic deep-water waves one may assume a surface elevation and a velocity potential of the form

$$\zeta(\boldsymbol{r}, t) = \zeta(\chi) \quad \text{and} \quad \phi(\boldsymbol{r}, z, t) = \phi(\chi, z),$$

where $\boldsymbol{r}$ is a two-dimensional (horizontal) position vector, z the vertical coordinate and

$$\chi = \boldsymbol{k}.\boldsymbol{r} - \sigma t \tag{2.2}$$

is a phase function. These may be substituted into an averaged Lagrangian

$$\mathscr{L} = -\frac{1}{2\pi}\int_0^{2\pi}\left\{\int_{-\infty}^{\zeta}\left[\frac{\partial\phi}{\partial t} + \tfrac{1}{2}(\nabla\phi)^2\right]\mathrm{d}z + \tfrac{1}{2}g\zeta^2\right\}\mathrm{d}\chi, \tag{2.3}$$

which is the obvious modification for deep water of Luke's (1967) Lagrangian. Equation (2.3) may be rewritten

$$\mathscr{L} = cI - T - V,$$

where I is the waves' impulse, or mean wave momentum, T is the mean kinetic energy, V is the mean potential energy density and c is the phase velocity, σ/k. Use of Levi-Civita's relation (see M.S.L.-H.)

$$2T = cI$$

gives

$$\mathscr{L} = T - V, \tag{2.4}$$

an expression which might alternatively be taken as defining $\mathscr{L}$.

In his table 2, M.S.L.-H. provides values of T, V, c and ak/π as functions of an appropriate expansion parameter. Hence $\mathscr{L}$ can be determined from equation (2.4). However, the results for c show that Lighthill's parameter z is an unsuitable variable since near the highest wave there are two solutions for each value of z. We have chosen to use a dimensionless variable proportional to the steepness squared, that is,

$$s = a^2k^2 = \pi^2H^2/\lambda^2, \tag{2.5}$$

where H and λ are the height and length of a wave. M.S.L.-H.'s expansion parameter would also be suitable, though it does not have such a simple physical interpretation.

The dispersion relation may be written

$$\sigma^2/gk = S(s), \tag{2.6}$$

where $S(s)$ is directly available from M.S.L.-H. Dimensional arguments give

$$\mathscr{L} = (\rho g/k^2)\, \mathscr{L}_2(a^2k^2, \sigma^2/gk), \tag{2.7}$$

which we write as
$$\mathscr{L} = (\rho g/k^2)\, \mathscr{L}_2[s, S(s)] = (\rho g/k^2)\, L(s), \tag{2.8}$$

where the newly defined function $L(s)$ is also available from M.S.L.-H.

In order to study the propagation of waves, expressions are needed for the wave-action density, $\mathscr{L}_\sigma$, and the wave-action flux, $-\mathscr{L}_k$. These may be obtained from the waves' mean energy density, $\mathscr{E}$, and its flux, $\mathscr{F}$, which may be found directly from their defining integrals, or following Whitham (1965), to be $\sigma\mathscr{L}_\sigma - \mathscr{L}$ and $-\sigma\mathscr{L}_k$ respectively. Thus the wave-action density,

$$\mathscr{L}_\sigma = A = (\mathscr{E}+\mathscr{L})/\sigma = 2T/\sigma, \tag{2.9}$$

and the wave-action flux,

$$-\mathscr{L}_k = \boldsymbol{B} = \mathscr{F}/\sigma = (3T-2V)\,\hat{\boldsymbol{k}}/k, \tag{2.10}$$

where $\hat{\boldsymbol{k}}$ is a unit vector parallel to k and the relations $\mathscr{L} = T-V$ and $|\mathscr{F}| = c(3T-2V)$, from M.S.L.-H., have been used.

In order to use M.S.L.-H.'s tables further dimensionless functions must be introduced. There are several functions one may choose to work with, e.g. dimensionless counterparts of T, V, A or B, but we choose $L(s)$, defined by (2.8), and $E(s)$, defined by

$$\mathscr{E} = T+V = (\rho g/k^2)\, E(s). \tag{2.11}$$

An advantage of the use of $L(s)$ is that the relatively inaccurate operation of finding the difference between T and V is performed directly on the figures in the table. Thus for wave-action and its flux we use

$$A = (\rho g/\sigma k^2)\,(E+L), \tag{2.12}$$

and

$$\boldsymbol{B} = (\rho g/2k^3)\,(E+5L)\,\hat{\boldsymbol{k}}. \tag{2.13}$$

The three functions $S(s)$, $L(s)$ and $E(s)$ are not independent. Consider $\mathscr{L}$ to be given by $\mathscr{L}_3(\sigma, k, s)$ and the dispersion equation by $\mathscr{L}_{3,s} = 0$. Then differentiation of the Lagrangian (2.8) and of $\mathscr{L}_3$, considering $\sigma = \sigma(k, s)$, gives

$$(\rho g/k^2)\, L'(s) = \mathscr{L}_{3,\sigma}\,\partial\sigma/\partial s. \tag{2.14}$$

Use of this equation in
$$\mathscr{E} = \sigma\mathscr{L}_\sigma - \mathscr{L} \tag{2.15}$$

gives
$$2S(s)\,L'(s) = [E(s)+L(s)]\,S'(s). \tag{2.16}$$

This relationship was checked satisfactorily by numerical integration. It corresponds to M.S.L.-H.'s equation (D), that is,

$$\mathrm{d}(T/c^2) = (1/c^2)\,\mathrm{d}V.$$

In the calculations described below the three functions $S(s)$, $L(s)$ and $E(s)$ were taken from M.S.L.-H.'s table 2. To use these functions at non-tabulated values of s a number of different interpolation and approximation methods were tested. Simple rational function approximations were found to be most convenient. These were found by a trial and error method. A limited number of selected tabulated values was used to determine each rational function. The error at the other tabulated values was evaluated and after a number of trials the functions given below were selected. They give results accurate to 4 decimal places except for the highest waves where the tabulated values are also less precise. (The recent results on highest waves by Longuet-Higgins & Fox (1978) mean that better approximations may now be available.)

$$S(s) = 1+s+\frac{2.6107s^2(0.1935-s)}{1-5.63543s+3.98484s^2}, \tag{2.17}$$

$$L(s) = \tfrac{1}{8}s^2-\frac{0.007157s^3}{1-6.73868s+9.64103s^2}, \tag{2.18}$$

$$E(s) = \tfrac{1}{2}s-\frac{0.19569s^2}{1-1.04488s-12.9792s^2}. \tag{2.19}$$

3. Steady waves on a current distribution

The problems studied in this paper are those of steady periodic waves on a current distribution which varies in one direction only. Choosing this to be the x direction we consider currents

$$\boldsymbol{U} = \boldsymbol{U}(x),$$

and suppose that this and all other quantities vary slowly with x. The continuity equation for the current is presumed to be satisfied by correspondingly small vertical velocity gradients which do not affect the waves to this approximation.

The equations which govern the waves' properties are as follows. The Doppler equation for wave frequency is

$$\omega = \sigma+\boldsymbol{k}\cdot\boldsymbol{U}, \tag{3.1}$$

where σ is the frequency of waves relative to the water and ω is their frequency in our reference frame. The total wave-action flux is $A\boldsymbol{U}+\boldsymbol{B}$ and from the wave-action equation we deduce that the wave-action flux in the x direction is constant, that is

$$(A\boldsymbol{U}+\boldsymbol{B})\cdot\boldsymbol{i} = \text{constant}. \tag{3.2}$$

The set of equations is completed by the dispersion equation

$$\sigma^2 = gkS(s), \tag{3.3}$$

and the wave-number consistency condition, which is

$$\nabla\wedge\boldsymbol{k} = 0 \tag{3.4}$$

for steady wave trains.

If we now introduce θ, the angle between $\boldsymbol{k}$ and $\boldsymbol{i}$, and let

$$\boldsymbol{U} = U(x)\,\boldsymbol{i}+V(x)\,\boldsymbol{j},$$

then equations (3.1), (3.2) and (3.4) become

$$\omega = \sigma + kU\cos\theta + kV\sin\theta, \tag{3.6}$$

$$(E+L)\,U/\sigma k^2 + \tfrac{1}{2}(E+5L)\cos\theta/k^3 = b, \tag{3.7}$$

and

$$k\sin\theta = m, \tag{3.8}$$

in which constants b and m are introduced and A and $\boldsymbol{B}$ have been replaced from equations (2.12) and (2.13). The constants b, m and ω are determined by initial conditions for the waves. It is often convenient to choose initial conditions corresponding to $\boldsymbol{U} = 0$, but there are other solution branches which do not correspond to any wave on still water.

For any specific current distribution $U(x)\,\boldsymbol{i} + V(x)\,\boldsymbol{j}$, the four unknown quantities k, σ, θ and s may be determined from the four equations (3.3, 6, 7 and 8). In practice, since x does not appear explicitly we do not need the functions $U(x)$, $V(x)$ and since the functions $S(s)$, $E(s)$ and $L(s)$ are complicated, it is simpler to choose a value of s and leave one of $U(x)$, $V(x)$ to be determined.

4. Waves on a current, $V(x)\,\boldsymbol{j}$

Waves propagating on a shearing current, $V(x)\,\boldsymbol{j}$, are refracted by any current gradient. The propagation of slowly varying, infinitesimal-amplitude waves may most easily be visualized in terms of rays, and if such rays become parallel to the current a caustic arises. It is of a simple symmetrical type, since waves reflected from the caustic are identical with those propagating toward the caustic at any point except for their changed direction. Details of linear solutions are given in § IIE of Peregrine (1976), with a slightly differing notation.

For finite-amplitude waves the concept of a ray path is not so readily available (see § 6). However in this case the simplicity of the situation makes solution straightforward. Equation (3.7) becomes

$$(E+5L)\cos\theta\sin^3\theta = 2m^3b, \tag{4.1}$$

after (3.8) is used. The solution depends on an initial choice of ω, m and b, then for any chosen s equation (4.1) may be solved for θ. Thereafter, k, σ and V are readily determined from (3.8), (3.3) and (3.6). There are two solutions for σ from equation (3.3), we have confined attention to positive values of σ and to positive values of ω. In this problem there is no appreciable loss of generality, since as noted in P. & S. for linear waves, the solutions are symmetrical about $V = \omega/m$.

Not only are the equations straight forward to solve but by eliminating k and σ they can be condensed to equation (4.1) and

$$(\omega - mV)^2 = gmS/\sin\theta. \tag{4.2}$$

From these two equations it is readily seen that solutions depend on a single function of position

$$[\omega - mV(x)]/(gm)^{\frac{1}{2}}, \tag{4.3}$$

and a single constant, m^3b. Thus once one set of solutions has been found, for given m^3b, all others are obtainable by translation and a change of scale.

The function (4.3) is not as easy to use as one would wish. For example, if ω and the initial value of θ are given, then m also depends on the initial wave steepness. We choose to show the behaviour of waves by using

$$v = V\omega/g$$

as a dimensionless function of position. The angle between the wave propagation direction and the current, $\phi = \frac{1}{2}\pi - \theta$, is also a more convenient variable since it is zero for linear waves at the caustic.

All waves are at their minimum steepness on this current distribution when $\phi = \frac{1}{6}\pi$. Figures 1 and 2 show properties of waves with initial condition $\phi = \frac{1}{6}\pi$ at $v = 0$. Changes in the direction of propagation of the waves with current are shown in figure 1. Each line is labelled with the value of s at $v = 0$. The linear solution is also shown. All the solution lines have a point where their gradient becomes singular, for a finite value of ϕ and v. Such a singularity implies that the assumption of a slowly varying wave train is invalid in its neighbourhood. This aspect of the solutions is discussed in more detail in P. & S., where this type of behaviour is called a type R caustic. Here, we note that the lines $b = 0$, other than that given by the linear solution, are $\phi = 0$ which meets the linear solution at the linear caustic point, and $\phi \pm \frac{1}{2}\pi$ (not shown) which are asymptotes of the linear solution as $k \to \infty$.

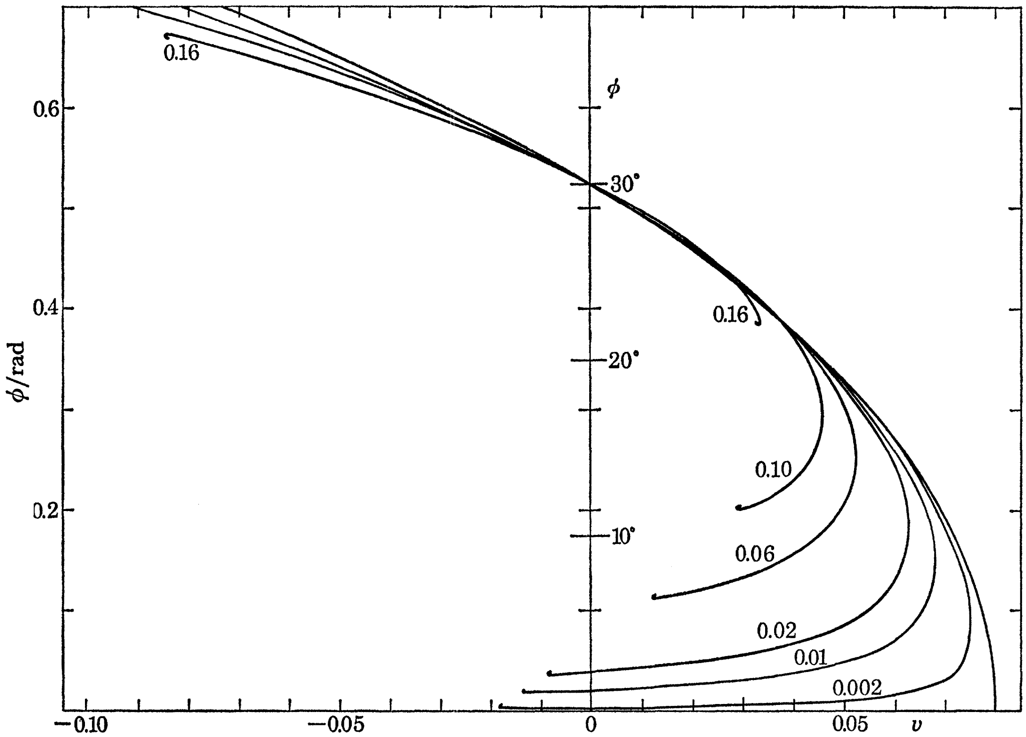

FIGURE 1. The angle between wave number vector and current, ϕ, for waves on a current $V(x)\boldsymbol{j}$ for different values of the initial steepness squared, $s = a^2k^2$, shown as functions of $v = V\omega/g$. All waves are taken to start on current $V = 0$ with $\phi = 30°$ and with the initial value of s indicated on each curve.

It is interesting to note that in *no* case is the maximum steepness reached before the singularity. Indeed, s can be quite small at the singular point, as may be seen from figure 2, which shows corresponding (v, s) lines. There seems little doubt that the singularity corresponds to reflexion, but the assumed slowly varying wave-train solution does not admit reflexion. However, there is a solution corresponding to reflexion, it is obtained by changing the sign of b, the total wave-action flux toward the caustic. This solution is readily obtained by reflecting the incident (v, ϕ) lines in the line $\phi = 0$ of figure 1, the (v, s) lines of figure 2 are unchanged.

Since the singularity may occur at small values of s it is a reasonable conjecture that no wave-breaking occurs.

If there are reflected waves then both incident and reflected wave trains will be at a small angle to each other and it is likely that nonlinear interactions will be significant. Hence there must be further doubt about the physical relevance of this solution in the neighbourhood of the singularity.

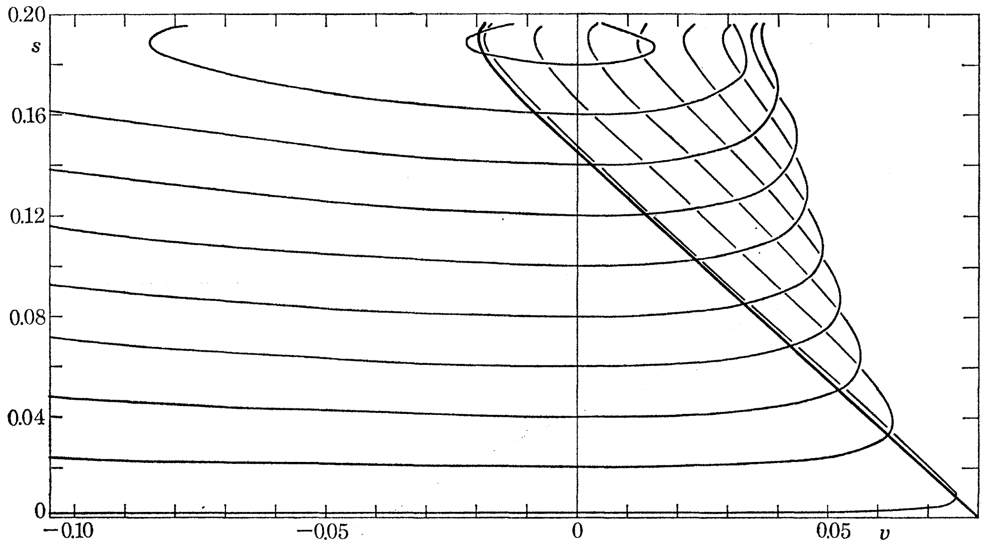

FIGURE 2. Variation of steepness squared, $s = a^2k^2$, for waves on a current $V(x)\boldsymbol{j}$ with $\phi = 30°$ at $V = 0$. Note the heavy line, for waves on the same current with $b = 0$, $\phi = 0$, which is the linear caustic.

No experiments describing waves at such a caustic have been described, but our observation of stationary waves on streams and rivers indicates that the caustics which bound waves trapped on a region of maximum current do not induce wave breaking.

Intermediate between the positive and negative b solutions there is the nonlinear solution for $b = 0$ with $\phi = 0$. This describes waves of constant wavelength with crests perpendicular to the current. The solution is very simple in this case since with $\theta = \frac{1}{2}\pi$ and k constant, equal to m, the dispersion relation (4.2) directly links s and V. That is, the wave steepness is just that required to give the appropriate phase velocity. This solution may be relevant to the actual caustic region if the velocity gradient is small enough. It is also relevant to waves propagating against a current and trapped by caustics each side of the current maximum, for example, the waves corresponding to the zero mode-number in the linear analysis of Peregrine & Smith (1975, § 4.2).

For the initial conditions given it is clear that the solution past the singular point on any line is irrelevant. However, initial conditions might be chosen to correspond to a solution point on the steeper side of the singularity. For example, in figure 3 the (v, ϕ) lines for waves of various steepnesses with $\phi = \frac{1}{12}\pi$ at $v = 0$ are shown. For waves with s_0 less than about 0.10 the behaviour is similar to that in figure 1 and qualitatively similar to the linear solution. On the other hand waves with $s_0 \geqslant 0.10$ exhibit different behaviour, they are refracted *away* from the current direction and decrease in steepness as v increases. They also approach the singularity we have just been interpreting as corresponding to reflexion from the 'other' direction. It should be possible to set up waves corresponding to this situation, but it requires further study, experimental or theoretical, before we know what would happen. However the fact that the $b = 0$ solution is close to this solution branch for small b is evidence that it has physical reality.

Each of the solution curves of figure 1, and the 'upper' branch of those in figure 3, also has a singularity when it is followed in the $-v$ direction. It occurs near the maximum of the waves' phase velocity and there seems little doubt that it represents wave breaking. As they approach this singularity the waves' crests are refracted more nearly parallel to the current and their wavelength shortens. For the linear solution $k \to \infty$. Such short-wave singularities occur with other current distributions (see next section) and do not seem to have been studied in any detail.

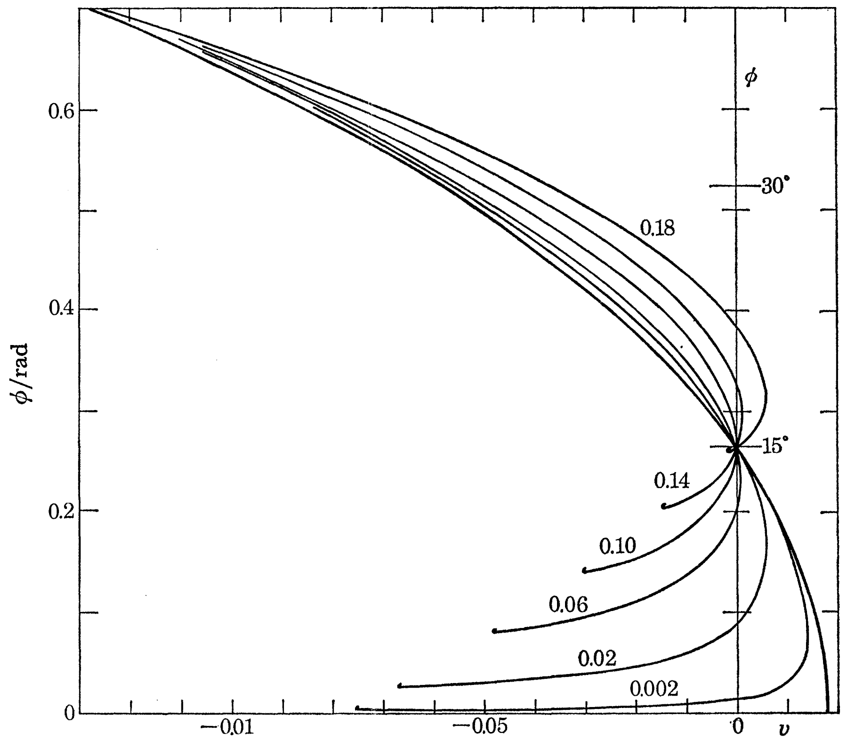

FIGURE 3. The angle between wave-number vector and current, ϕ, for waves on a current $V(x)\boldsymbol{j}$ for different values of the initial steepness squared, $s = a^2k^2$, shown as functions of $v = V\omega/g$. All waves are taken to start on current $V = 0$ with $\phi = 15°$ and with the initial value of s indicated on each curve.

5. Waves on a current, $U(x)\,\boldsymbol{i}$

The current distribution envisaged here is unidirectional and slowly varying in magnitude, as in an idealized river of varying depth. Only waves travelling directly with or against the current are considered, we take $\boldsymbol{k} = k\boldsymbol{i}$, although there is no intrinsic difficulty in studying more general cases. Consideration of positive and negative values of U gives all the solutions. However, we only report results for non-negative values of ω. For k positive this corresponds to omitting those solutions where waves propagate against a current but with a phase velocity relative to the water which is less than the current velocity. Such solutions have been calculated.

There are two values of the velocity at which the linear theory is singular. One is $U = -g/4\omega$, the 'stopping' velocity at which the waves' group velocity relative to the water is equal and opposite to the current. This is an example of a caustic since the rays of linear theory are smooth lines in (x, t) space, all touching the position where $U(x) = -g/4\omega$. There is a solution branch corresponding to reflected waves which has a further, 'short-wave', singularity at $U = 0$. More details of the linear solution are given by Peregrine (1976, § IID).

To solve this example it is convenient to introduce $c = \sigma/k$, the phase velocity of the waves relative to the water. Equations (3.6) and (3.3) become

$$\omega = k(c+U), \tag{5.1}$$

and

$$c^2 k = gS(s). \tag{5.2}$$

With these equations it is easy to eliminate k and U from (3.7) to give

$$\omega(E+L)\, c^7 - \tfrac{1}{2}gS(E-3L)\, c^6 = bg^4S^4. \tag{5.3}$$

(Note that in making this equation dimensionless one may choose between using $c\omega/g$ or $c|\omega|/g$ and some care is needed if $\omega < 0$.) Solutions are found by choosing s and then solving the polynomial (5.3) for c. Then k and U are determined from (5.2) and (5.1) respectively.

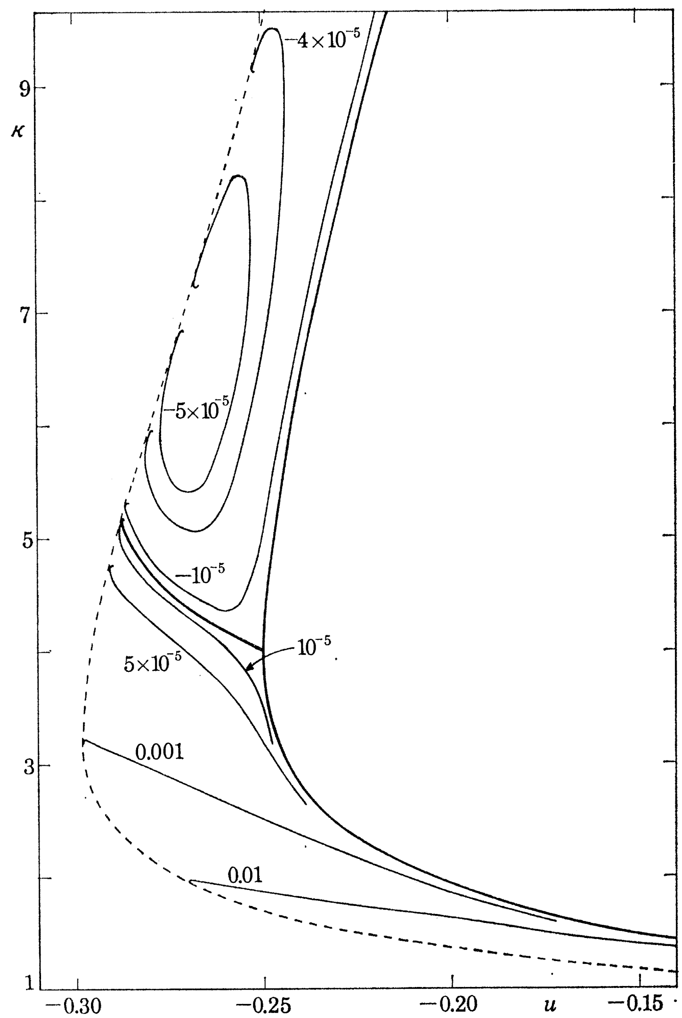

FIGURE 4. Variation of dimensionless wave number $\kappa = kg/\omega^2$ for waves on a current $U(x)\mathbf{i}$ for different values of the wave-action flux shown as functions of $u = U\omega/g$. The heavy lines correspond to $B = 0$. The numbers on the other full lines correspond to the value of s at $U = 0$. Negative values correspond to reflected waves. The dotted line indicates the boundary of maximum phase velocity, equation (5.4).

There are completely different solutions for corresponding positive and negative values of b. That is, waves which may be incident on or reflected from a caustic. Solutions given by $b = 0$ are for waves 'stopped' by an adverse current.

Solutions for $\omega > 0$ are illustrated by their trajectories in the (κ, u) plane in figure 4, where

$$\kappa = kg/\omega^2, \quad \text{and} \quad u = U\omega/g.$$

Each line is identified by the value of s on zero current, except where b is negative a negative value of s_0 has been given to show which incident solution it corresponds to. The area shown in figure 4 concentrates on the region near the linear stopping velocity, $u = -\frac{1}{4}$, since finite-amplitude effects are most significant there. The two lines of zero wave-action flux, $b = 0$,

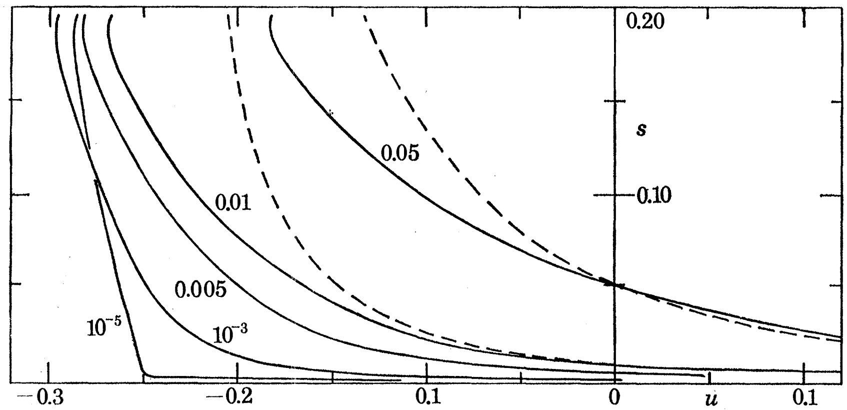

FIGURE 5. Variation of steepness squared, $s = a^2k^2$, for waves on a current $U(x)\,\boldsymbol{i}$. Initial values of s are shown on each line. $u = U\omega/g$. The dashed lines show solutions obtained by using linear theory.

which are the infinitesimal solution and the stopped-waves solution are shown. These and the line corresponding to maximum wave properties outline the two solution regions. The line of maximum phase velocity is readily found from equations (5.1) and (5.2) to be

$$(1-\kappa u)^2 = \kappa S_{\text{max}}. \tag{5.4}$$

The solutions in the region where waves may have propagated from water at rest, that is those in the lower part of figure 4, are rather as one would expect. They diverge from the linear solution, with a longer wavelength and smaller steepness on any given adverse current, and if the adverse current is strong enough they reach breaking point, with no other singularity intervening. Examples of the variation of steepness squared, s, with current are given in figure 5. The maximum adverse current needed to make a given wave train break may be greater or less than the linear stopping velocity, $-g/4\omega$, depending on the constants ω and b defined by the waves. The maximum adverse current against which waves may make headway is $gS_{\text{max}}/4\omega = 0.299g/\omega$.

Breaking can dissipate a large proportion of wave energy, but at the same time wave momentum is transferred to the water and leads to a current distribution that varies with depth. The resulting vorticity is swept into the incident waves by the current and a full description should include such effects. Clearly the present solutions do not contribute significantly to the understanding of a region of breaking waves.

If waves propagate to a point with the linear stopping velocity and still have a relatively small steepness than reflexion must be expected. After all, the linear wave theory does provide approximate solutions with total reflexion in this neighbourhood. Such waves can most readily occur in practice if they are created on a current which has a velocity close to the stopping velocity. Examples include waves generated by a boat progressing up a river (Peregrine 1971) and surface waves riding on long internal waves (Gargett & Hughes 1972).

Strictly, if there is reflexion solutions corresponding to two wave trains should be examined. However, if one thinks in terms of a long group of waves, it is reasonable to examine the solutions corresponding to reflected waves. By conservation of wave action (Hayes 1970) the incident and reflected wave trains have the same magnitude for their wave-action flux, but their wave action travels in opposite directions. For example, a typical solution with small steepness

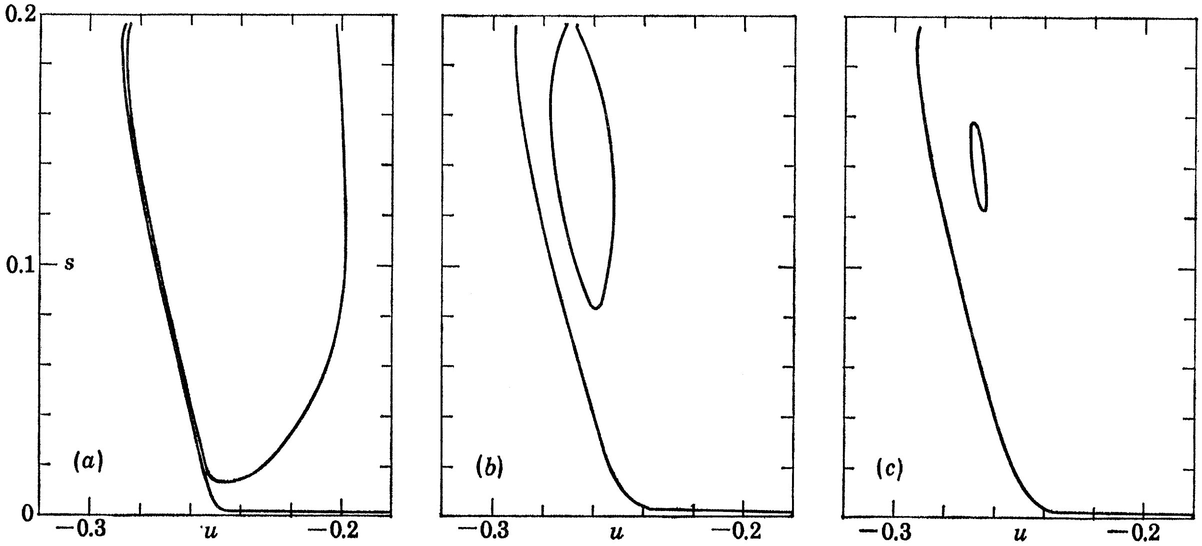

FIGURE 6. Variation of steepness squared, $s = a^2k^2$, for waves on a current $U(x)\,\boldsymbol{i}$. $u = U\omega/g$. S_0 = (a) 10^{-5}, (b) 5×10^{-5}, (c) 5.9×10^{-5}. Reflected waves are included.

at the linear stopping velocity is that for $s_0 = 10^{-5}$. Solutions for κ and for s are shown in figures 4 and 6*a*. It is easy to believe that reflexion does occur and that the part of the solution where incident and reflected waves are both close to the zero wave-action flux solution is not relevant. On the other hand the finite-amplitude solution is quite regular and there is no indication that it may be inappropriate. However, if a solution were found that allowed for both wave trains some such indication might be found. This caustic region is an example of an S type caustic in the classification of P. & S.

The solutions for waves reflected from a stopping point, or more generally, for waves created with negative values of b and positive ω have surprising properties. If they are swept onto weaker, less adverse, currents a singularity occurs in the slowly varying solution. See figures 4 and 6*a*, *b*. Superficially it looks similar to the caustic singularity described in the previous section: both $\mathrm{d}\kappa/\mathrm{d}u$ and $\mathrm{d}s/\mathrm{d}u$ become infinite for finite values of κ and s. The value of s at this singularity, typically 0.12, is not near its maximum. Another surprising feature is that these solutions do not exist for $s_0 < -6\times10^{-5}$. It seems unlikely that the solution shown in figure 6*c* for $s_0 = -5.9\times10^{-5}$ has any direct physical interpretation.

This strange singularity can not be physically similar to a caustic singularity since there is no solution branch corresponding to further wave reflexion, though there is the, rather distant,

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solution corresponding to incident waves. It seems unlikely to directly represent breaking since the steepness is not especially large. One possibility is that energy is transferred to free higher harmonic waves. Whalin (1971) reported substantial energy transfer to free second harmonics in an experiment with a focal region. Another possibility is that there may be some other mechanism of transfer of energy and momentum to the mean flow which depends on the gradient of the current.

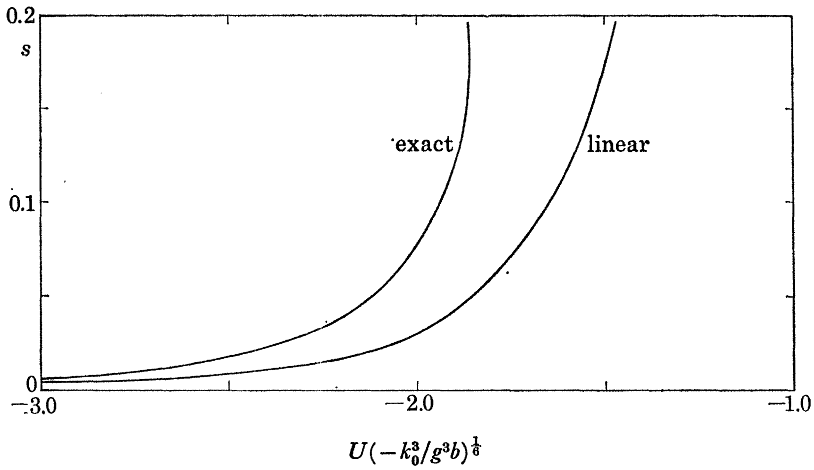

FIGURE 7. Variation of steepness squared, $s = a^2k^2$ for stationary waves on a current $U(x)\,\boldsymbol{i}$. Exact and linear theory.

The singularity does have its counterpart in the linear solution; as $U \to 0-$ so $k \to \infty$. Similar, $k \to \infty$ singularities also occur in the linear solutions for $\omega = 0$ and $\omega < 0$. The case $\omega = 0$ is particularly simple, it corresponds to stationary waves, i.e. $U + c = 0$, such as may be generated by an obstacle in a stream. The solution is readily obtained from the wave-action equation (3.7), which can be reduced to

$$bk^3 = -\tfrac{1}{2}k_0^3(E-3L), \tag{5.5}$$

where k_0 is the initial value of k, and the dispersion relation

$$U^2 = gS/k. \tag{5.6}$$

The solution for s as a function of $U(-k_0^3/g^3b)^{\frac{1}{6}}$ is given in figure 7, together with the corresponding linear solution. In this case the finite-amplitude solution also shows a rapid variation with U, but the actual singularity is at sufficiently large s that we may, reasonably associate it with wave breaking. The solutions for $\omega < 0$ resemble that for $\omega = 0$.

There seems to have been no attempt to find an 'exact' linear solution in the neighbourhood of a '$k \to \infty$' singularity. A short investigation shows that there are exact linear solutions with singularities as $U \to 0$, but there is also the case of regular standing waves studied by Taylor (1962).

6. Group velocity

The concept of group velocity is very valuable in understanding and predicting the propagation of linear waves; see, for example, Lighthill (1965*a*). Its extension to nonlinear waves presents difficulties since there are many possible 'group velocities'. Some of these are defined and discussed below.

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A direct physical approach to the problem of wave propagation leads to a velocity for the propagation of some integral quantity, Q say; where we define a velocity

$$\boldsymbol{c}_Q = \frac{\text{flux of } Q}{\text{density of } Q}. \tag{6.1}$$

Note that the suffix Q on a velocity $\boldsymbol{c}$ does not denote differentiation as it does elsewhere. The most obvious quantities to consider are energy, and wave action. For deep-water waves these give

$$\boldsymbol{c}_E = \frac{\mathscr{F}}{\mathscr{E}} = \frac{\sigma(E+5L)}{2kE}\,\hat{\boldsymbol{k}}, \tag{6.2}$$

$$\boldsymbol{c}_A = \frac{\boldsymbol{B}}{A} = \frac{\sigma(E+5L)}{2k(E+L)}\,\hat{\boldsymbol{k}}. \tag{6.3}$$

It is interesting to note that it is not generally possible to define a propagation velocity for momentum. For example, we might look for $\boldsymbol{c}_I$ such that

$$(\boldsymbol{c}_I)_\alpha I_\beta = S_{\alpha\beta}, \tag{6.4}$$

where I_α is the momentum density vector and $S_{\alpha\beta}$ the momentum flux tensor. It is readily seen that for deep water waves, where

$$I_\alpha = (2T/c)\,\hat{k}_\alpha, \tag{6.5}$$

and

$$S_{\alpha\beta} = (T-V)\,\delta_{\alpha\beta} + (3T-2V)\,\hat{k}_\alpha \hat{k}_\beta, \tag{6.6}$$

then no solution exists for $\boldsymbol{c}_I$. This latter result (6.6) is found from the expression for finite depth

$$S_{\alpha\beta} = \int_{-h}^{\zeta} \left\{p\delta_{\alpha\beta} + \rho\,\frac{\partial\phi}{\partial x_\alpha}\,\frac{\partial\phi}{\partial x_\beta}\right\} \mathrm{d}z - \tfrac{1}{2}gh^2\,\delta_{\alpha\beta},$$

and the results derived in M.S.L.-H., § 3.

The velocities $\boldsymbol{c}_E$ and $\boldsymbol{c}_A$ are clearly not equal. Their departure from the linear group velocity is shown in figure 8 as a function of s. The importance of wave-action in slowly varying wave theory suggests that $\boldsymbol{c}_A$ may be significant. For example

$$\boldsymbol{c}_A + U = 0, \tag{6.7}$$

gives the zero wave-action flux line in figure 4. However, figure 4 shows clearly that the maximum adverse current on which a stopped wave train can exist is less than the maximum adverse current against which waves can propagate (e.g. the waves for $s_0 = 0.001$). This latter velocity, from equation (5.4) is $u = -\frac{1}{4}S_{\max}$ which bears no obvious relation to $\boldsymbol{c}_A$ or $\boldsymbol{c}_E$.

Velocities defined as in (6.1) suffer from the disadvantage that they are based on the properties of a *uniform* wave train. Even for linear waves there is a possible ambiguity in a steady wave field. Longuet-Higgins (1964) points out that the energy-flux vector is arbitrary in that any vector field $\boldsymbol{F}$ satisfying $\nabla\cdot\boldsymbol{F} = 0$ may be added to the chosen vector. This comment can apply to other conserved quantities. As a further example consider the problems of determining the appropriate expression for the momentum density of electromagnetic waves in matter described in Robinson (1973, pp. 96–99) and resolved by Jones (1978) and Jones & Leslie (1978).

Another relatively simple approach is to take the linear relation

$$\boldsymbol{c}_g = \sigma_{\boldsymbol{k}} \tag{6.8}$$

and extend it to nonlinear waves. Now, σ depends not only on $\boldsymbol{k}$ but on some measure of the amplitude. The actual value of σ_k depends on which amplitude measure is kept constant. Lighthill (1965*a*) suggested $\mathscr{L}/\sigma$ which gives it the value $\boldsymbol{c}_E$ of (6.2). Willebrand (1975) gives an explicit expression, his equation (25), for near-linear waves which includes a term showing how it varies with different measures of amplitude. Willebrand examines the effect of a spectrum of waves on propagation velocities. He shows that a sufficiently smooth spectrum leads to unique propagation velocities for its component waves in the near-linear approximation.

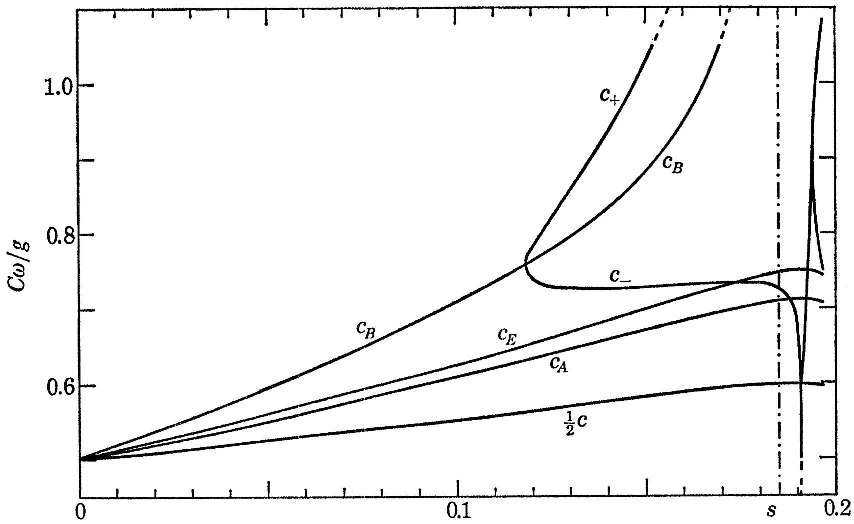

FIGURE 8. Possible extensions of group velocity for deep water waves. The ratio of the velocity to the velocity of linear waves of the same frequency is plotted for the wave-action velocity c_A, the wave-energy velocity c_E, the basic group velocity c_B and the characteristic velocities c_+ and c_-. Note, the dot–dash line is at the steepness where c_B and c_+ are singular.

This arises since in the limiting process from a finite sum of waves to a continuous spectrum the self-interaction terms (i.e. the nonlinear effects considered here) become negligible compared with the interactions between different components. The actual expression obtained is complicated.

Propagation velocities may be obtained from the equations describing slowly varying waves. The velocities of the characteristics of the equations are the propagation velocities of small modulations of a wave train. Whitham (1967) and Lighthill (1967) first derived them for water waves, but Hayes (1973) gives a more complete account. Hayes defines a 'basic group velocity' $\boldsymbol{c}_B$, by

$$\boldsymbol{c}_B = \boldsymbol{B}_A \tag{6.9}$$

where in this definition $\boldsymbol{B}$ is the wave-action flux considered as a function of $(\boldsymbol{k}, A; \boldsymbol{x}, t)$, that is after eliminating σ and using A as the amplitude measure. Hayes writes the Lagrangian in the form

$$\mathscr{L} = \sigma A - \mathscr{H}(\boldsymbol{k}, A). \tag{6.10}$$

After some manipulation we find

$$\left.\frac{\partial \boldsymbol{B}}{\partial A}\right|_k = \mathscr{H}_{kA} = \frac{\sigma[E'(s)+5L'(s)]}{2kE'(s)}\,\hat{\boldsymbol{k}}, \tag{6.11}$$

which also equals

$$\mathscr{H}_{Ak} = \partial\sigma/\partial\boldsymbol{k}|_A. \tag{6.12}$$

The basic group velocity is also the mean of the characteristic velocities for the time-dependent equations for slowly varying waves (see Hayes 1973, § 4 or Whitham 1974, § 15.2). These equations describe modulations of both amplitude and wavenumber for a slowly varying wave train. Hayes calls the characteristic velocities 'basic signal velocities'. In this case there are two velocities, $\boldsymbol{c}_+$ and $\boldsymbol{c}_-$, such that

$$\boldsymbol{c}_\pm = \mathscr{H}_{Ak} \pm \mathscr{H}_{AA}\, \mathscr{H}_{kk}\cdot\boldsymbol{n}/[\mathscr{H}_{AA}\, \mathscr{H}_{kk}:(\boldsymbol{nn})]^{\frac{1}{2}}, \tag{6.13}$$

where $\boldsymbol{n}$ is a unit vector normal to the *perturbation* (or modulation) wave fronts.

Since
$$\mathscr{H}_{AA}\, \mathscr{H}_{kk} = \frac{gS'(E+5L)}{4kS^{\frac{1}{2}}E'}\left\{\boldsymbol{I} - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}\left[4 - \frac{5(E'+5L')\,(E+L)}{2(E+5L)\,E'}\right]\right\}, \tag{6.14}$$

we note that
$$\mathscr{H}_{AA}\, \mathscr{H}_{kk}:(\boldsymbol{nn}) \tag{6.15}$$

may be either positive or negative and the velocities $\boldsymbol{c}_\pm$ may be real or complex. The mean velocity $\boldsymbol{c}_B$, and $\boldsymbol{c}_\pm$ where they are real are shown in figure 8 for $\boldsymbol{n}$ parallel to $\boldsymbol{k}$. It should be noted that $\boldsymbol{c}_+$ and $\boldsymbol{c}_B$ become singular when $E'(s) = 0$ at the dimensionless energy maximum. Thus for the highest waves these velocities have little physical value.

The complex values of $\boldsymbol{c}_\pm$ correspond to the partial differential equations being elliptic, in which case an initial-value problem is not well posed, and a uniform wave train is unstable to small modulations. This aspect is discussed by Lighthill (1965*b*, 1967) and details of the instability are described by Benjamin (1967) and Lake *et al.* (1977).

Hayes (1973) gives the stability boundary, based on this criterion, for near-linear waves in all depths of water. Using the result (6.14) extends this stability analysis to finite-amplitude deep-water waves. The sign of the expression (6.15) is the same as that of

$$1 - \cos^2\psi[4 - \tfrac{5}{2}(E'+5L')\,(E+L)/(E+5L)\,E'], \tag{6.16}$$

where ψ is the angle between the wavenumber vector and the modulation wave-number vector, that is

$$\hat{\boldsymbol{k}}\cdot\boldsymbol{n} = \cos\psi. \tag{6.17}$$

The stability boundary in the (ψ, s) plane is shown in figure 9. For $\boldsymbol{n}$ parallel to $\boldsymbol{k}$ this result, $s = 0.118$, may be compared with the result, $H/\lambda = 0.108$ or $s = 0.115$, obtained by Lighthill (1967) with his approximate Lagrangian. Longuet-Higgins (1978*a*, *b*) gives results of direct calculations of the stability of a uniform wave train. These show that Benjamin-Feir instability ceases for sufficiently steep waves at a value of $ak = 0.347$ (that is $s = 0.120$) for very long modulations. There is also a stability boundary which varies between $S'(s) = 0$ and $E'(s) = 0$, that is $0.185 \leqslant s \leqslant 0.191$. It does not seem to be directly related to the instability Longuet-Higgins (1978*b*) finds at $ak = 0.41$, $s = 0.168$.

The rate of growth of an unstable modulation is most readily found by considering the modulation to be of the form

$$\exp\,[\mathrm{i}k_1(\boldsymbol{x}\cdot\boldsymbol{n} - c_r t) + k_1 c_i t]. \tag{6.18}$$

Since the modulations are non-dispersive

$$\boldsymbol{c}_\pm\cdot\boldsymbol{n} = c_r \pm \mathrm{i}c_i. \tag{6.19}$$

The actual rate of growth depends linearly on the wave number of the modulation. It thus increases for shorter modulations and hence the maximum growth rate cannot be determined from this 'long-modulation' approach. However, contours of c_i/c_r are shown in figure 9.

The instability of most deep-water wave trains, which leads to waves with substantial modulations, makes much of the analysis presented here seem rather academic. However, such modulations have mainly been studied theoretically for near-linear waves, and for near-linear waves approaching a caustic Smith (1976) finds stability, which gives cause to hope that this work is indeed relevant to physical problems. Furthermore, Lake *et al.* (1977) describe experiments on deep-water wave trains which become strongly modulated but which then return to being a uniform wave train. The near-linear theory is shown to describe the experiments with one exception.

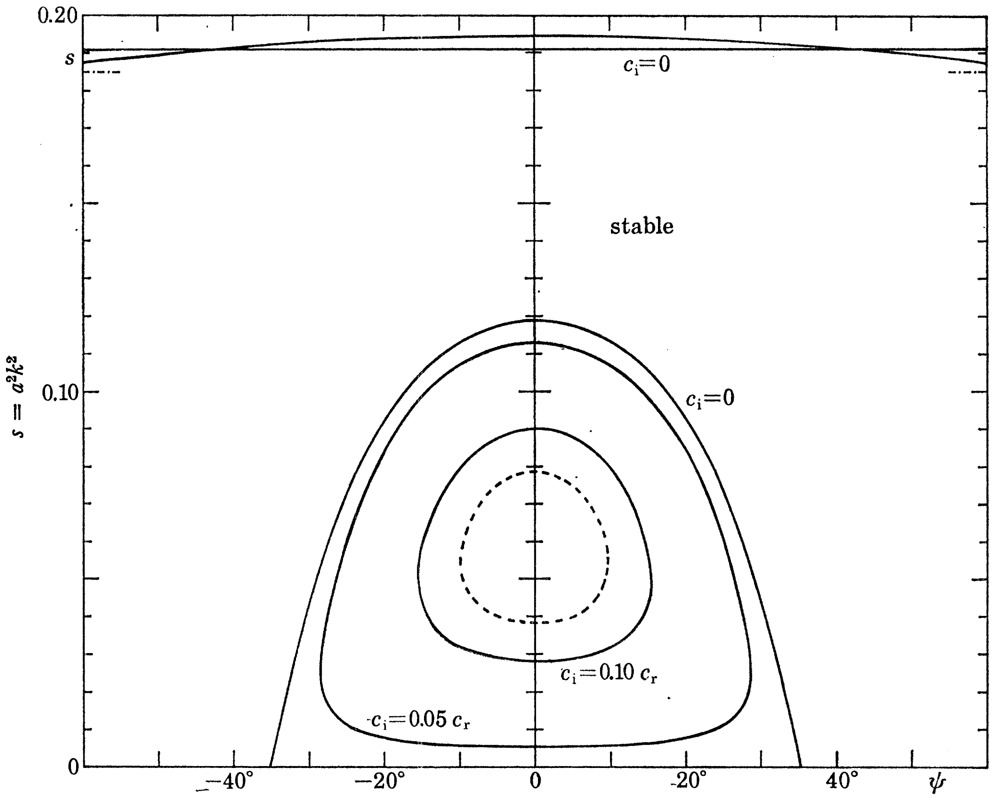

FIGURE 9. Stability boundary in the (ψ, s) plane for deep-water waves subject to long modulations at an angle ψ. Within the lower unstable region contours of c_i/c_r are shown at values 0.05, 0.10 and 0.11. The line $E'(s) = 0$ is shown as ·–·–·–·, and the upper stability boundary meets it at $\psi = 90°$.

Numerical results and experiments show a return to the initial state except when 'wave trains with large initial steepness' were generated. In this case the experiments lead to a recurrence of a uniform wave train, but with a longer period than the initial wave train. In every such case 'there were waves which were modulated to the point of capillary generation or breaking'. This implies that some waves were sufficiently steep to pass the stability boundary $(s = 0.118)$ shown in figure 9. The nonlinear Schrödinger equation used to describe the wave behaviour theoretically is very unlikely to be adequate in such circumstances since it is a near-linear approximation. Even terms up to $O(s^3)$, i.e. $O(a^6k^6)$, are insufficient to give a good estimate of the stability boundary.

For unidirectional propagation of deep-water waves the appropriate near-linear Schrödinger equation in a frame of reference moving at the linear group velocity is (from Hasimoto & Ono 1972, or Davey 1972, or by the methods of P. & S., § 7),

$$\mathrm{i}\,\frac{\partial a}{\partial t}-\frac{\sigma_0}{8k_0^2}\,\frac{\partial^2 a}{\partial x^2}-\tfrac{1}{2}\sigma_0 k_0^2|a^2|a = 0, \tag{6.20}$$

where the wave amplitude is $\mathrm{Re}\,\{a(x,\,t)\,\exp\mathrm{i}(kx-\sigma_0 t)\}$. This equation has solutions corresponding to an isolated wave group:

$$a = a_0\,\mathrm{sech}^2\,\{2^{\frac{1}{2}}k_0^2 a_0(x-Vt)\}\,\exp\,\{-\tfrac{1}{2}\mathrm{i}k_0^2\sigma_0 a_0 t-(4\mathrm{i}k_0^2 V/\sigma)\,(x-Vt)\}. \tag{6.21}$$

To satisfy approximations made in deriving (6.20), $k_0 V/\sigma_0$ must be small. Indeed V can be set equal to zero with little loss of generality since it only corresponds to the modulation of a wave train with the slightly different wave number,

$$k = k_0(1-4k_0 V_0/\sigma_0). \tag{6.22}$$

The remaining term in the exponential gives the first correction to the dispersion equation for finite amplitude. However, the propagation speed of the wave envelope is unaffected; it is just the linear group velocity.

7. Conclusions

The results of this paper illustrate two markedly different types of caustic. The companion paper, P. & S., shows that these two types are typical for near linear waves, and names them R and S types.

The R type caustic of § 4 has a singularity in the approximation presented here, but all the indications are that it is a *regular* type. That is, actual water waves do not have singularities (wave breaking) unless the wave steepness becomes close to its maximum. The mathematical singularity is at a smaller steepness and near-linear analysis such as in § 7 of P. & S. is likely to give a uniform solution.

On the other hand, the S type of caustic of § 5 is *singular* for a wide range of solutions. That is solutions presented here are consistent with the approximations made right up to a steepness which one can readily associate with wave breaking. Although linear and near-linear approximations are found in P. & S. which correspond to reflexion without any singularity. It is likely that they only apply to very gentle waves. This is exemplified by the lack of any reflected wave solution for sufficiently large values of the wave-action flux in the finite-amplitude solution, and is the most puzzling aspect of the analysis. It seems as if it may be associated with the singularities which all the solutions have for waves being swept onto still water where the linear solution gives $k\to\infty$.

The propagation of waves against adverse currents naturally leads one to consider group velocity. Various possible extensions of the concept to nonlinear waves are discussed in the previous section. It shows, for deep-water waves, the difficulties of such an extension.

There are several possible extensions of this work. It can be easily applied to wave systems with circular symmetry and a brief paper giving results is in preparation. Caustic problems will not be fully solved until solutions involving both incident and reflected waves are analysed. This may require careful consideration of any depth-varying currents generated by the reflexion

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of the waves' momentum. Indeed, it is probably simpler, and more valuable, to solve corresponding problems for waves in water of finite, and varying, depth using Cokelet's (1977) accurate solutions.

A preliminary account of this work was presented at the I.U.T.A.M. symposium on 'Waves in water of variable depth' at Canberra, July 1976 (Peregrine & Thomas 1976).

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